`dense_general_position.tex` compiled November 11, 2006
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In this note we will see that there exists a countably infinite set X of points in the plane $\mathbb{R}^2$ such that any (nondegenerate) circle, no matter how small or where it is, contains some point $x \in X$ and yet no three points in X are in a straight line. At first glance this may seem a little surprising since any large point set has a huge number of "forbidden lines" formed by looking at any pair of points therein — hence one might intuitively guess that it is eventually impossible to miss all those lines.

These two properties are traditionally referred to as *density* and being *in general position:* specifically, a set $X \subset \mathbb{R}^2$ is called **dense** iff $\forall y \in \mathbb{R}^2, \epsilon > 0$, $\exists x \in X : ||x - y|| < \epsilon$, where $||z||$ is the Euclidean norm $\sqrt{z_1^2 + z_2^2}$ when $z = (z_1, z_2)$. The set $X \subset \mathbb{R}^2$ is **in general position** iff $\forall a, b, c \in X$, the matrix

$$M = \begin{pmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{pmatrix}$$

is nonsingular. This last assertion about a, b, c is a fancy (and rigorous) way of saying that they are not in a straight line. If all three points were on the line $y = mx + d$, then

$$M \begin{pmatrix} m \\ -1 \\ d \end{pmatrix} = 0,$$

showing that M would then be singular; there is another case when they're all on a vertical line as well. Similarly, whenever $Mv = 0$ for a nonzero vector v, we can show that a, b, c obey a nice linear equation.

Theorem 1 *There exists a countably infinite set $X \subset \mathbb{R}^2$ which is both dense and in general position.*

We will see two proofs. There is a common element to both - we need to use a countable sequence $\mathcal{S}$ of sets $s_1, s_2, \ldots$ with $s_i \subset \mathbb{R}^2$ such that for any circle $B_\epsilon(y)$ ($\epsilon > 0$) there exists some $s_i \subset B_\epsilon(y)$; here $B_\epsilon(y)$ denotes the circle of radius ϵ centered at point $y \in \mathbb{R}^2$.

How do we know such a sequence exists? There are many ways to see this. One way: start by noticing that the rationals $\mathbb{Q}$ are countable, and simply list out all rational points $z \in \mathbb{Q}^2$ and rational radii $\delta \in \mathbb{Q}$, making each corresponding $B_\delta(z)$ a set $s_i \in \mathcal{S}$. So, given an arbitrary circle $B_\epsilon(y)$, we just need to choose a rational $\delta < \epsilon/2$ and rational z with $||z - y|| < \epsilon/2$, and we're guaranteed that $B_\epsilon(y) \supset B_\delta(z) \in \mathcal{S}$ as desired.

Proof 1.

First we'll see a probabilistic proof.

Here's the short version of this proof: pick a random set of points and almost surely they'll fit the conditions of the theorem. But maybe we should fill in some details.

Choose some probability distribution P on $\mathbb{R}^2$ so that any set X with zero area (a null set) has $P(X) = 0$ and every set s_i in $\mathcal{S}$ has $P(s_i) > 0$. Next choose, in an i.i.d. fashion, a random countably infinite set $X \in \mathbb{R}^2$ using P. We claim that, with probability 1 (i.e. "almost surely"), this set is dense and in general position.

Let's check that it is almost surely dense. Let X_n denote the set of the first n points chosen in X. Then clearly $P(X_n \cap s_i = \emptyset) = (1-p)^n$ where $p = P(s_i)$. Next $P(X \cap s_i = \emptyset) = \lim_{n\to\infty}(1-p)^n = 0$. To cover the case of multiple s_i simultaneously, we can define the event

$$E_i \equiv (X \cap s_i = \emptyset),$$

and $E \equiv \cup_i E_i$. We have seen that $P(E_i) = 0$ for each i. Then $P(E) = P(\cup E_i) \le \sum_i P(E_i) = 0$, so that almost surely X hits each of the sets s_i. For any circle $B_\epsilon(y)$ there is some $s_i \in \mathcal{S}$ with $s_i \subset B_\epsilon(y)$. Thus $P(E) = 0$ suffices to show that almost surely every circle in the plane is hit, giving the density of X.

Now let's check that X is almost surely in general position. Define the event

$$F_n \equiv (X_n \text{ is in general position}),$$

and let L_n denote the set of lines through any two points in X_n. Since L_n is a finite collection of lines, it has zero area (it's a null set); so $P(x \in L_n) = 0$. We'll see that $P(F_n) = 1$ for all n by induction on n. It's obvious for $n = 1$ (basis case) since there aren't even 3 points to be in a line! Next assume $P(F_n) = 1$, and

$$P(F_{n+1}|F_n) = P(x \notin L_n|F_n) = \frac{P(x \notin L_n \& F_n)}{P(F_n)}$$
$$= P(x \notin L_n) = 1 - P(x \in L_n) = 1.$$

Finally, since $P(F_{n+1}) = P(F_{n+1}|F_n)P(F_n)$ and $P(F_n) = 1$, we get that $P(F_{n+1}) = 1$ as well. Define event

$$F = \cap_n F_n,$$

which means precisely that X is in general position, and notice that $F_n \searrow F$, that is, when $m \ge n$, event F_m can only be true if F_n is also true. This means

$$P(F) = \lim_{n\to\infty} P(F_n) = 1,$$

and we have accomplished our goal. □

Proof 2. This proof is a derandomized version of the last.

Now that we've set up most of the framework already, it's easy. We'll build $X = \{x_1, x_2, \ldots\}$ one point at a time. Given X_n just choose any x_{n+1} which does not hit any lines in L_n and is in s_{n+1}. We can always choose such a point since $s_{n+1} - L_n$ has positive area and hence is clearly nonempty.

As we saw in proof 1, X is dense as soon as every set s_i is hit, which is certainly the case here. In addition, each set X_n is in general position so that the entire collection X is also. (If X were not in general position then there must exist some finite set X_n which is also not.)

Hence our less-randomly-constructed set X also satisfies the theorem. $\square$